Fractional Turán's theorem and bounds for the chromatic number

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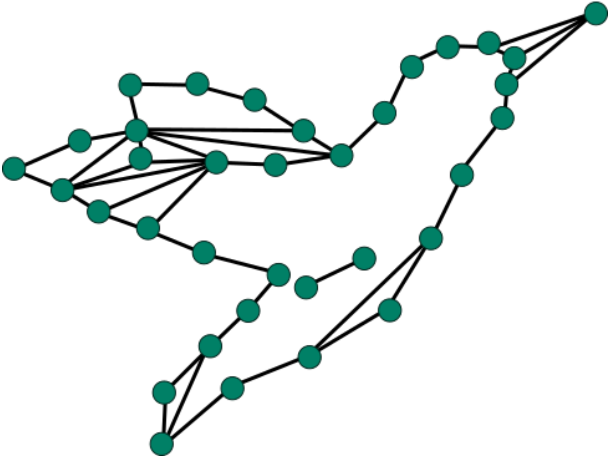
Graphs

- A *graph* is a pair of sets $G = (V, E)$
- V is called the set of *vertices*.
- E is called the set of *edges*. The elements from E are some pairs of vertices.

Hummingbird Graph



Cliques and clique number

- A subset of vertices is a *clique* if there is an edge between any two of them.
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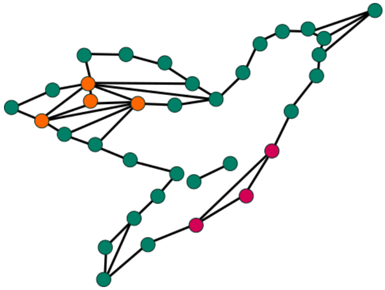

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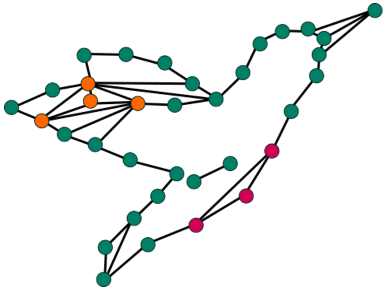

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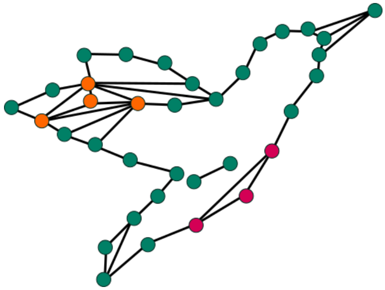

- How are $|E|$ and $\omega(G)$ related?
- **Intuition:** If $|V|$ is fixed and $|E|$ is large, then $\omega(G)$ is large.

Mantel's theorem and Turán's theorem

Theorem (Mantel, 1907)

Let $G = (V, E)$ *be a graph such that* $|V(G)| = n$. *If*

$$|E(G)| > \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil,$$

then

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Theorem (Turán, 1941)

Let $G = (V, E)$ *be a graph such that* $|V(G)| = n$ *and* r *a positive integer. If*

$$|E(G)| > \frac{r-1}{r} \cdot \frac{n^2}{2},$$

then

$$\omega(G) \geq r + 1.$$

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edges.
- **Intuition:** A large *proportion* of the maximum possible edges yields a large clique of *fixed size* with respect to n.
- **Question:** Does there exist a result that given a large proportion of edges guarantees that $\omega(G) \geq cn$?
- **Question:** Does there exist a result that given a large proportion of edges guarantees that $\omega(G) \geq f(n)$, with f a function such that $f(n) \to \infty$?

Interval graphs

- In general, they do not exist. Turán's theorem is best possible. Therefore, there are graphs such that

$$|E(G)| > 0.9999 \cdot \frac{n^2}{2} \quad \text{and} \quad \omega(G) < 0.0001 \cdot n.$$

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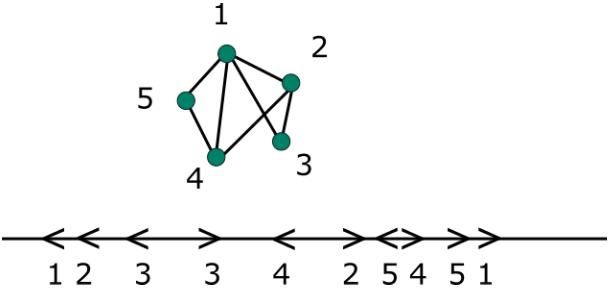
- What happens if we restrict ourselves to some families of graphs?
- *Interval graphs:* $G = (V, E)$, where V is a finite family of bounded real intervals and two intervals form an edge if they intersect.
- We will use $\mathcal{G}_I$ to denote the family of all interval graphs.

Example of an interval graph



Another example of an interval graph

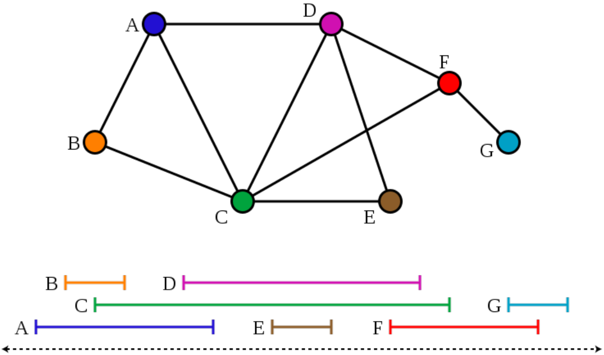


Figure: By David Eppstein, Public Domain

A theorem by Katchalski and Liu

Theorem (Katchalski, Liu, 1979)

Let $G \in \mathcal{G}_I$ be an interval graph on n vertices and $\alpha \in (0,1)$ a real number. If G has more than

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- Why can we have a better Turán's type result? In this case, it is because of the geometry.

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- In which families can we have a Turán's type result that guarantees $\omega(G) \geq c \cdot n$?
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- Can we find some nice applications in geometry or other areas of mathematics?

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- A graph G is *bipartite* if $\chi(G) \leq 2$.

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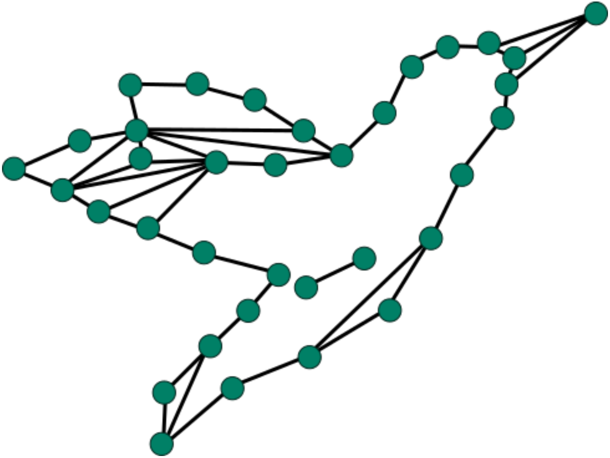

Proper coloring of the Hummingbird Graph

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- Families in which $\chi(G') \leq f(\omega(G'))$ for any induced subgraph G' of a graph G in the family are interesting and have been widely studied (Gyarfas, 1987). For example, *perfect graphs*.

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- How much do we need $|V(G)|$?

Proposition

For any graph G, the following inequality holds:

$$\chi(G) \leq \frac{1}{2} \cdot |V(G)| + \frac{1}{2} \cdot \omega(G).$$

Proof

- Let G be a graph, $n = V(G)$, $\chi = \chi(G)$ and $\omega = \omega(G)$. We color G using χ colors.
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$$\begin{aligned} n &= a_1 + 2a_2 + 3a_3 + \ldots + ra_r \\ &\geq a_1 + 2(a_2 + a_3 + \ldots + a_r) \\ &= a_1 + 2(\chi - a_1) = 2\chi - a_1. \end{aligned}$$

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- Therefore $\chi \leq \frac{n+a_1}{2}$. We now need to prove $a_1 \leq \omega$.

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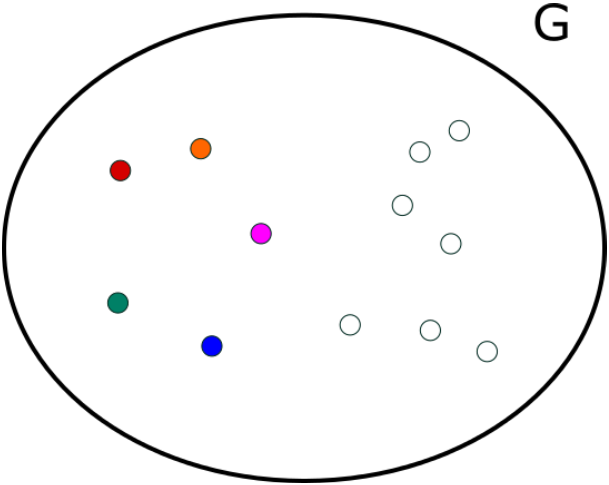

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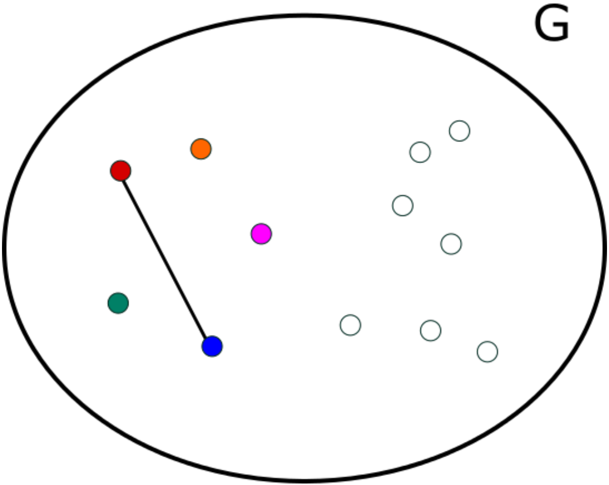

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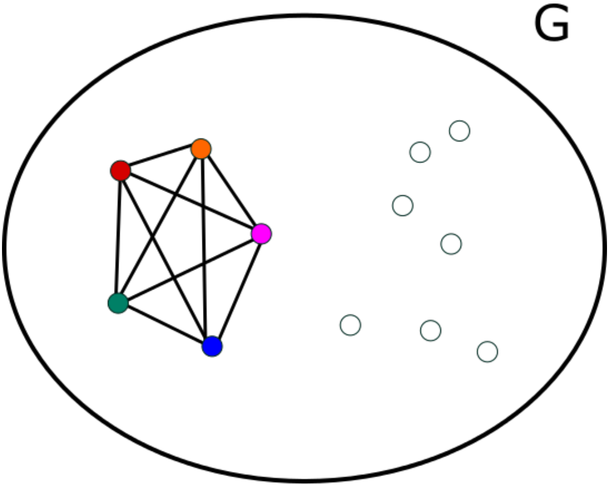

The results: Some equivalences

Theorem (L.M. and L. Montejano, 2014)

Let $\mathfrak{G}$ be a family of graphs that is closed under induced subgraphs. Then the following three statements are equivalent:

- *There are real numbers c and d such that*
  - *for every graph $G \in \mathfrak{G}$ we have $\chi(G) \leq c\omega(G)$ and*
  - *for every $B \in \mathfrak{G}$, B bipartite, we have $|E(B)| \leq d|V(B)|$.*
- *$\mathfrak{G}$ has a linear fractional Turán-type theorem, namely, there exists β such that*
  - *for $G \in \mathfrak{G}$ if $|E(G)| \geq \alpha\binom{|V(G)|}{2}$, then $\omega(G) \geq \alpha\beta n$.*
- *There exists a constant C such that*
  - *If G is a graph on n vertices such that $\omega(G) \leq k$, then $|E(G)| \leq Cnk$.*

The results: Chromatic number bound

Theorem (L.M. and L. Montejano, 2014)

For any $\epsilon > 0$ there exists a function f_ϵ such that for any graph G the following inequality holds:

$$\chi(G) \leq \epsilon \cdot |V(G)| + f_\epsilon(\omega(G)).$$

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Theorem (L.M. and L. Montejano, 2014)

Let $\mathfrak{G}$ be a cuis family of graphs in which $|E(B)| \leq d|V(B)|$ for a global constant d.
Then for any $\alpha > 0$, the graphs in the set $\{G \in \mathfrak{G} : |E(G)| \geq \alpha\binom{|V(G)|}{2}\}$ satisfy that $\omega(G) \to \infty$ as $|V(G)| \to \infty$.

Some problems

- Is it true that for any graph G we have the following?

$$\chi(G) \leq \frac{1}{3} \cdot |V(G)| + 1000 \cdot \omega(G).$$

- Is is true if we change 1000 by a larger constant?

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Thank you for your attention!