Points defining triangles with distinct circumradii

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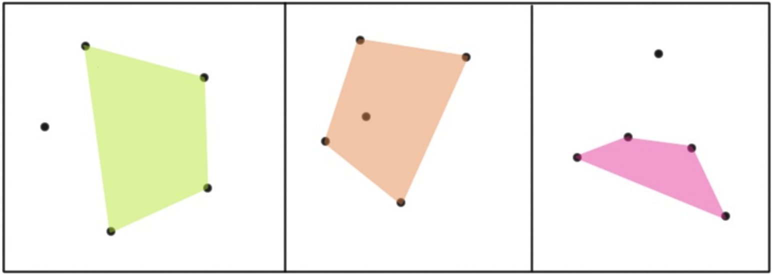
Five points

(E. Klein) Among any 5 points in general position on the plane, there are always 4 of them in convex position.



Happy Ending Theorem

Theorem

For every positive integer k there exists a number n_k such that if we take n_k or more points on the plane in general position, then we can find k of them in convex position.

Known bounds:

$$1 + 2^{k-2} \leq n_k \leq \binom{2k-5}{k-2} = \mathcal{O}\left(\frac{4^k}{\sqrt{k}}\right).$$

Convex position $\rightarrow$ Distinct circumradii

Question on distinct circumradii

In Austral. Math. Soc. Gaz. 1975, Erdős asks:

Problem

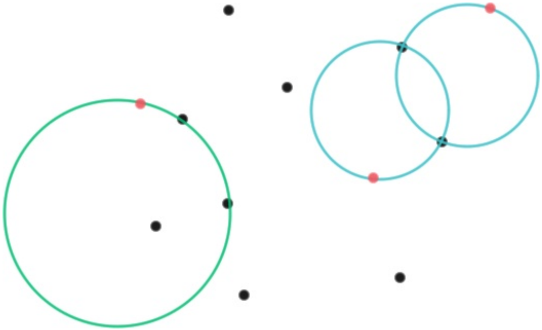
Let k be a positive integer. Is it true that there always exists an integer n_k such that in every set of n_k points on the plane in general position (no 3 on a line or 4 on a circle) we can find a set of k of them such that all the triangles they define have distinct circumradii?

Three years later he claims to have an affirmative answer for $n_k = 2\binom{k-1}{2}\binom{k-1}{3} + k$. But he inadvertently left out a non-trivial case.

Erdős argument

- Take n points on the plane and G a *maximal* good set. Suppose $|G| = \ell$. Let r_1, ..., $r_{\binom{\ell}{3}}$ be the distinct circumradii.
- (*) Any other point lies in a circle of radius r_i that goes through two of the points of G.
- Therefore, by the general position hypothesis $n - \ell \leq 2\binom{\ell}{2}\binom{\ell}{3}$.

Erdős argument



The theorems

Theorem

(L.M. and E. Roldán, 2014)

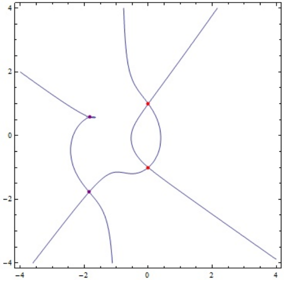
- $n_4 \leq 9$
- $n_5 \leq 37$

Theorem

(L.M. and E. Roldán 2014) There exists a number $n_k = \mathcal{O}(k^9)$ *such that for every* n_k *points in general position we can find k of them with distinct circumradii.*

New idea

- For $\{A, B\}$ y $\{C, D\}$ distinct pairs of points, we consider the set of points X such that $R(ABX) = R(CDX)$. We call it $\mathcal{C}(AB, CD)$.
- $\mathcal{C}(AB, CD)$ is an algebraic curve of degree at most 6.



Sketch of the proof

- We bound n_4 and n_5.
- We prove a $\mathcal{O}(n^5)$ for when all the points lie on an algebraic curve.
  - Maximal set
  - Bezout's theorem + (n_4) + (n_5)
- We prove the main theorem.
  - Maximal set
  - $\mathcal{O}(n^5)$ result for algebraic curves

References

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