Snakes, lattice path matroids and a conjecture by Merino and Welsh

Leonardo Ignacio Martínez Sandoval

Ben-Gurion University of the Negev
Joint work with
Kolja Knauer - Université Aix Marseille
Jorge Ramírez Alfonsín - Université de Montpellier

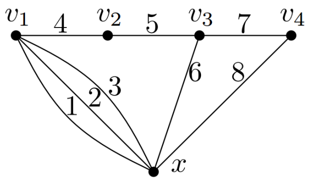
October 11
DCG-DIOSPT Seminar, EPFL, Lausanne

Some ideas in graph theory

Let G be a graph and label its edges.

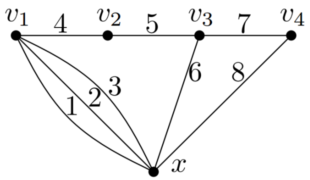
Some ideas in graph theory

Let G be a graph and label its edges.



Some ideas in graph theory

Let G be a graph and label its edges.



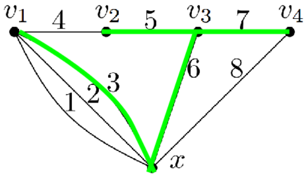
We are interested the number of *spanning trees* $\tau(G)$, of *acyclic orientations* $\alpha(G)$ and of *totally cyclic orientations* $\alpha^*(G)$.

Spanning trees

Connected an acyclic subgraphs of G whose edges cover all the vertices.

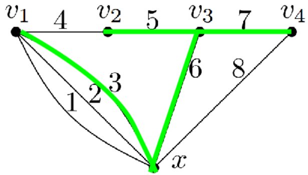
Spanning trees

Connected an acyclic subgraphs of G whose edges cover all the vertices.



Spanning trees

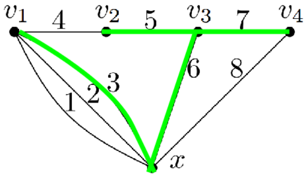
Connected an acyclic subgraphs of G whose edges cover all the vertices.



We use given labels to denote the trees. In the picture we see the spanning tree $\{3, 5, 6, 7\}$. Other examples: $\{2, 5, 7, 8\}$ and $\{1, 4, 5, 8\}$.

Spanning trees

Connected an acyclic subgraphs of G whose edges cover all the vertices.



We use given labels to denote the trees. In the picture we see the spanning tree $\{3, 5, 6, 7\}$. Other examples: $\{2, 5, 7, 8\}$ and $\{1, 4, 5, 8\}$. There are 27 in total i.e. $\tau(G) = 27$.

Spanning trees

The set $\mathcal{T}$ *of spanning trees* satisfies:

Spanning trees

The set $\mathcal{T}$ *of spanning trees* satisfies:

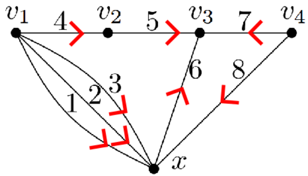
1. $\mathcal{T} \neq \emptyset$
2. If A and B are elements in $\mathcal{T}$ and we take an element $a \in A \setminus B$, then we can find $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is in $\mathcal{T}$.

Acyclic orientations

We assign a direction to each edge avoiding oriented cycles.

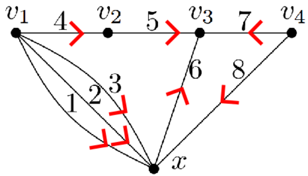
Acyclic orientations

We assign a direction to each edge avoiding oriented cycles.



Acyclic orientations

We assign a direction to each edge avoiding oriented cycles.



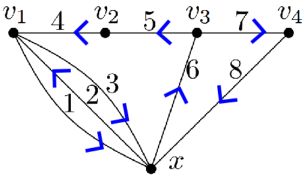
The value of $\alpha(G)$ is 42.

Totally cyclic orientations

We assign a directions so that each edge lies in an oriented cycle.

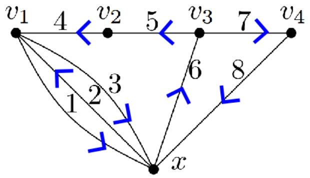
Totally cyclic orientations

We assign a directions so that each edge lies in an oriented cycle.



Totally cyclic orientations

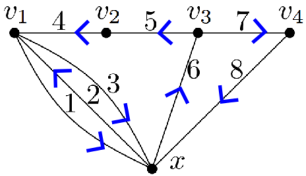
We assign a directions so that each edge lies in an oriented cycle.



The value of $\alpha^*(G)$ is also 42.

Totally cyclic orientations

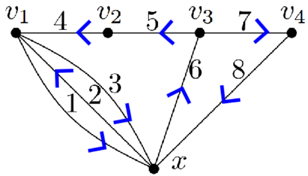
We assign a directions so that each edge lies in an oriented cycle.



The value of $\alpha^*(G)$ is also 42. Coincidence

Totally cyclic orientations

We assign a directions so that each edge lies in an oriented cycle.



The value of $\alpha^*(G)$ is also 42. Coincidence?

Merino-Welsh conjectures

Note that in the example above

$$\max(\alpha, \alpha^*) = \max\{42, 42\} \geq 27 = \tau.$$

Merino-Welsh conjectures

Note that in the example above

$$\max(\alpha, \alpha^*) = \max\{42, 42\} \geq 27 = \tau.$$

In 1999, Criel Merino and Dominic Welsh noted that $\alpha \geq \tau$ for some families of graphs, and when this was not the case, $\alpha^* \geq \tau$. From here they conjectured:

Merino-Welsh conjectures

Note that in the example above

$$\max(\alpha, \alpha^*) = \max\{42, 42\} \geq 27 = \tau.$$

In 1999, Criel Merino and Dominic Welsh noted that $\alpha \geq \tau$ for some families of graphs, and when this was not the case, $\alpha^* \geq \tau$. From here they conjectured:

Conjecture

For every 2-connected and loopless graph G

$$\max\left(\alpha(G), \alpha^*(G)\right) \geq \tau(G).$$

Merino-Welsh conjectures

Later on (2009), Conde and Welsh propose stronger but easier to handle versions of the conjecture:

Merino-Welsh conjectures

Later on (2009), Conde and Welsh propose stronger but easier to handle versions of the conjecture:

Conjecture

For any 2-connected and loopless graph we have:

1. *(Additive)* $\alpha(G) + \alpha^*(G) \geq 2 \cdot \tau(G)$.
2. *(Multiplicative)* $\alpha(G) \cdot \alpha^*(G) \geq \tau(G)^2$.

Merino-Welsh conjectures

Later on (2009), Conde and Welsh propose stronger but easier to handle versions of the conjecture:

Conjecture

For any 2-connected and loopless graph we have:

1. *(Additive)* $\alpha(G) + \alpha^*(G) \geq 2 \cdot \tau(G)$.
2. *(Multiplicative)* $\alpha(G) \cdot \alpha^*(G) \geq \tau(G)^2$.

$$\max(\alpha, \alpha^*) \geq \frac{\alpha + \alpha^*}{2} \geq \sqrt{\alpha \cdot \alpha^*}.$$

Partial results

The conjecture is still open, but there has been constant progress towards its solution

- 1999 - Merino, Welsh - The conjecture is posed and solved for some families of graphs
- 2009 - Conde, Merino - Threshold graphs, bipartite graphs, $9,945,269$ cases verified by computer

Partial results

The conjecture is still open, but there has been constant progress towards its solution

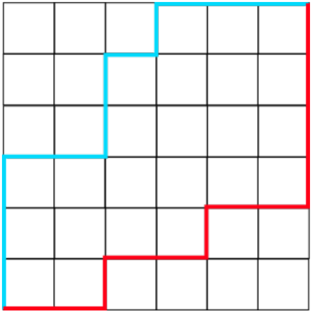
- 1999 - Merino, Welsh - The conjecture is posed and solved for some families of graphs
- 2009 - Conde, Merino - Threshold graphs, bipartite graphs, $9,945,269$ cases verified by computer
- 2010 - Thomassen - G with at least $4n$ edges or at most $\frac{16n}{15}$ edges, multigraphs with maximum degree 3 and planar triangulations
- 2011 - Chávez-Lomelí, Merino, Noble, Ramírez-Ibáñez - Wheels, whirls, 3-regular graphs with girth at least 5, complete graphs
- 2014 - Noble, Royle - Series parallel graphs

Lattice path matroids

- Let m and n be positive integers. We draw a $m \times n$ board.
- *Lower path P* and *upper path Q* (they can intersect, but not cross).

Lattice path matroids

- Let m and n be positive integers. We draw a $m \times n$ board.
- *Lower path P* and *upper path Q* (they can intersect, but not cross).

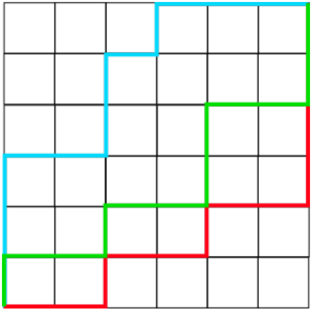


Lattice paths

- We consider all the *lattice paths* that lie between P and Q.

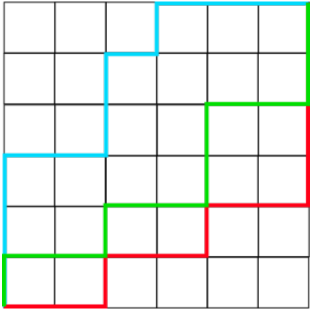
Lattice paths

- We consider all the *lattice paths* that lie between P and Q.



Lattice paths

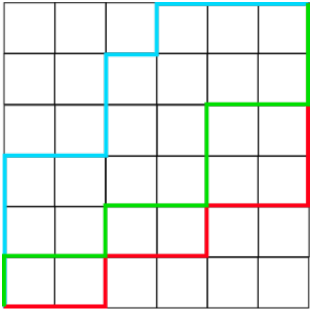
- We consider all the *lattice paths* that lie between P and Q.



We can name them according to in which steps they go "up".

Lattice paths

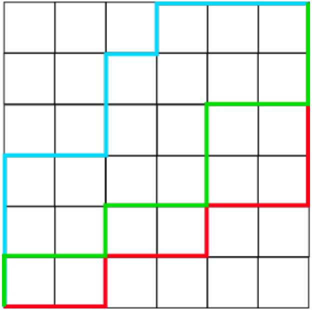
- We consider all the *lattice paths* that lie between P and Q.



We can name them according to in which steps they go "up".
$\{1,4,7,8,11,12\}$.

Lattice paths

- We consider all the *lattice paths* that lie between P and Q.



We can name them according to in which steps they go "up". $\{1,4,7,8,11,12\}$. Another one: $\{1,2,3,6,7,11\}$.

Lattice paths

The set $\mathcal{B}$ *of valid lattice paths* satisfies the following:

Lattice paths

The set *$\mathcal{B}$ of valid lattice paths* satisfies the following:

1. $\mathcal{B} \neq \emptyset$.
2. If A and B are in $\mathcal{B}$ and we take an element $a \in A \setminus B$, then we can find an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is in $\mathcal{B}$.

Lattice paths

The set *$\mathcal{B}$ of valid lattice paths* satisfies the following:

1. $\mathcal{B} \neq \emptyset$.
2. If A and B are in $\mathcal{B}$ and we take an element $a \in A \setminus B$, then we can find an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is in $\mathcal{B}$.

These are the same properties satisfied by the spanning trees of a graph.

Matroids

A *matroid* is a structure formed by a *ground set* E and a set $\mathcal{B} \subseteq \mathcal{P}(E)$ of *bases*, for which 1. and 2. hold

Matroids

A *matroid* is a structure formed by a *ground set E* and a set $\mathcal{B} \subseteq \mathcal{P}(E)$ of *bases*, for which 1. and 2. hold

We have seen two ways of constructing matroids

Matroids

A *matroid* is a structure formed by a *ground set* E and a set $\mathcal{B} \subseteq \mathcal{P}(E)$ of *bases*, for which 1. and 2. hold
We have seen two ways of constructing matroids

- From the spanning trees of a graph: *graphic matroids*.

Matroids

A *matroid* is a structure formed by a *ground set E* and a set $\mathcal{B} \subseteq \mathcal{P}(E)$ of *bases*, for which 1. and 2. hold
We have seen two ways of constructing matroids

- From the spanning trees of a graph: *graphic matroids*.
- From valid lattice paths in a board: *lattice path matroids (LPMs)* (2013 - Bonin, de Mier, Noy).

Another way to construct matroids

- We take a vector space V over the field $\mathbb{F}$.

Another way to construct matroids

- We take a vector space V over the field $\mathbb{F}$.
- We fix a set of vectors $S = \{v_1, v_2, \ldots, v_j\}$.

Another way to construct matroids

- We take a vector space V over the field $\mathbb{F}$.
- We fix a set of vectors $S = \{v_1, v_2, \dots, v_j\}$.
- We set as ground set $[j]$ and $I \subseteq [j]$ is a base if $\{v_i : i \in I\}$ is a vector base for $\mathrm{span}(S)$.

Another way to construct matroids

- We take a vector space V over the field $\mathbb{F}$.
- We fix a set of vectors $S = \{v_1, v_2, \ldots, v_j\}$.
- We set as ground set $[j]$ and $I \subseteq [j]$ is a base if $\{v_i : i \in I\}$ is a vector base for $\operatorname{span}(S)$.

These matroids are called *representable over* $\mathbb{F}$.

Another way to construct matroids

- We take a vector space V over the field $\mathbb{F}$.
- We fix a set of vectors $S = \{v_1, v_2, \ldots, v_j\}$.
- We set as ground set $[j]$ and $I \subseteq [j]$ is a base if $\{v_i : i \in I\}$ is a vector base for $\mathrm{span}(S)$.

These matroids are called *representable over $\mathbb{F}$*. If $\mathbb{F}$ is $GF(2)$, we simply say that the matroid is *binary*.

Independent sets

- Any subset of a base set is called an *independent set*.

Independent sets

- Any subset of a base set is called an *independent set*.
- For a subset A a subset of E we define $r(A)$ (the *rank of A*) as the size of the largest independent set contained in A.

Independent sets

- Any subset of a base set is called an *independent set*.
- For a subset A a subset of E we define $r(A)$ (the *rank of A*) as the size of the largest independent set contained in A.
- A powerful invariant for matroids is the *Tutte polynomial*. It is a two variable polynomial in x and y defined as follows:

Independent sets

- Any subset of a base set is called an *independent set*.
- For a subset A a subset of E we define $r(A)$ (the *rank of A*) as the size of the largest independent set contained in A.
- A powerful invariant for matroids is the *Tutte polynomial*. It is a two variable polynomial in x and y defined as follows:

$$T(M; x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Tutte polynomial

For LPMs, $T(M;1,1) =$ number of valid lattice paths

Tutte polynomial

For LPMs, $T(M; 1, 1) =$ number of valid lattice paths
For graphic matroids:

- $T(M; 1, 1) = \#$ of spanning trees
- $T(M; 2, 0) = \#$ of acyclic orientations
- $T(M; 0, 2) = \#$ of totally cyclic orientations

Tutte polynomial

For LPMs, $T(M;1,1) =$ number of valid lattice paths
For graphic matroids:

- $T(M;1,1) = \#$ of spanning trees
- $T(M;2,0) = \#$ of acyclic orientations
- $T(M;0,2) = \#$ of totally cyclic orientations

The Tutte polynomial can be obtained recursively from contraction and deletion operations.

Merino-Welsh conjectures

Conjecture (Merino-Welsh conjectures: matroid versions)

Let M be a matroid without loops or coloops and T_M its Tutte polynomial. Then:

1. $\max(T_M(2,0), T_M(0,2)) \geq T_M(1,1)$.
2. *(Additive)* $T_M(2,0) + T_M(0,2) \geq 2 \cdot T_M(1,1)$.
3. *(Multiplicative)* $T_M(2,0) \cdot T_M(0,2) \geq T_M(1,1)^2$.

Partial results

- Graphic matroids corresponding to the previously mentioned families of graphs

Partial results

- Graphic matroids corresponding to the previously mentioned families of graphs
- 2011 - Chávez-Lomelí, Merino, Noble, Ramírez-Ibáñez - Catalan matroids and paving matroids

Partial results

- Graphic matroids corresponding to the previously mentioned families of graphs
- 2011 - Chávez-Lomelí, Merino, Noble, Ramírez-Ibáñez - Catalan matroids and paving matroids
- 2015 - Knauer, M-S, Ramírez-Alfonsín - LPMs

Partial results

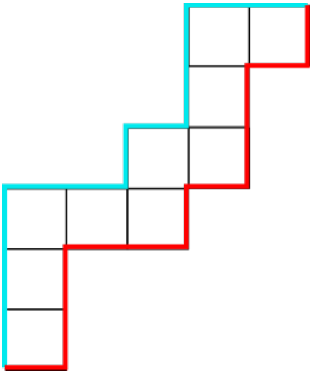
- Graphic matroids corresponding to the previously mentioned families of graphs
- 2011 - Chávez-Lomelí, Merino, Noble, Ramírez-Ibáñez - Catalan matroids and paving matroids
- 2015 - Knauer, M-S, Ramírez-Alfonsín - LPMs + sharpened version and equality cases

Snakes

If P and Q surround a 1-width strip, we call the LPM a *snake*.

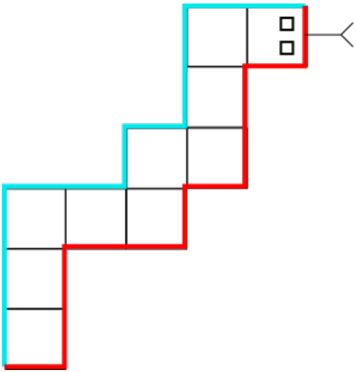
Snakes

If P and Q surround a 1-width strip, we call the LPM a *snake*.



Snakes

If P and Q surround a 1-width strip, we call the LPM a *snake*.

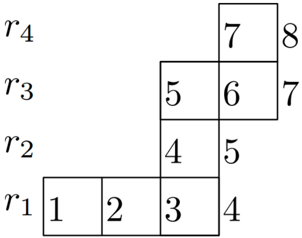


Valid lattice paths for snakes

Consider the following snake:

Valid lattice paths for snakes

Consider the following snake:



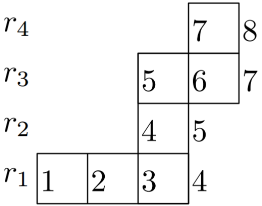
A valid lattice path is $\{3, 5, 6, 7\}$. Two more are $\{2, 5, 7, 8\}$ y $\{1, 4, 5, 8\}$

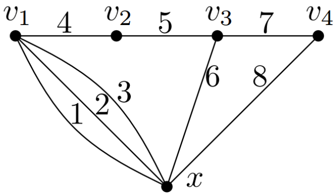
Equivalent structures

There is a key connection between snakes and a special family of graphs

Equivalent structures

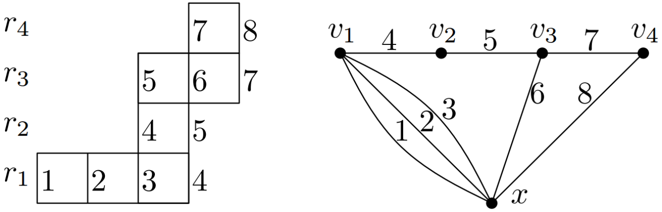
There is a key connection between snakes and a special family of graphs





Equivalent structures

There is a key connection between snakes and a special family of graphs



In this case, we have a graphic matroid of a *generalized fan*.

Results: Characterization of snakes

Not every LPM is graphic.

Results: Characterization of snakes

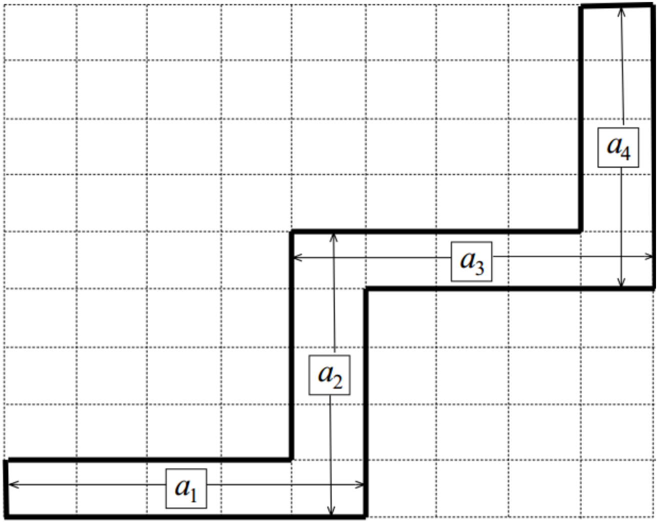
Not every LPM is graphic. Which are?

Theorem (Characterization of snakes)

Given a connected LPM M the following statements are equivalent:

- *M is a snake*
- *M is graphic*
- *M is a graphic matroid of a generalized fan*
- *M is binary*

Results: Notation for snakes



Results: Explicit formulas for snakes

Let $Fib(n)$ be the set of all n-digit binary sequences $b = (b_1, \ldots, b_n)$ without adjacent ones.

Results: Explicit formulas for snakes

Let *Fib*(n) be the set of all n-digit binary sequences $b = (b_1, \ldots, b_n)$ without adjacent ones.

Proposition

The number of valid lattice paths for the snake $S(a_1, a_2, \ldots, a_n)$ *is*

$$\sum_{b \in Fib(n+1)} \prod_{i=1}^{n} (a_i - 1)^{1-|b_{i+1}-b_i|}.$$

Results: Explicit formulas for snakes

Let $Fib(n)$ be the set of all n-digit binary sequences $b = (b_1, \ldots, b_n)$ without adjacent ones.

Proposition

The number of valid lattice paths for the snake $S(a_1, a_2, \ldots, a_n)$ is

$$\sum_{b \in Fib(n+1)} \prod_{i=1}^{n} (a_i - 1)^{1-|b_{i+1}-b_i|}.$$

Proposition

The value of $\alpha \cdot \alpha^*$ for the snake $S(a_1, a_2, \ldots, a_n)$ is

$$4 \cdot \prod_{i=1}^{n} (2^{a_i} - 1).$$

Results: Merino-Welsh conjecture for LPMs

Theorem

Let M be a loopless and coloopless LPM that is not the direct sum of trivial snakes. Then

$$T_M(2,0) \cdot T_M(0,2) \geq \frac{4}{3} \cdot T_M(1,1)^2$$

Results: Merino-Welsh conjecture for LPMs

Theorem

Let M be a loopless and coloopless LPM that is not the direct sum of trivial snakes. Then

$$T_M(2,0) \cdot T_M(0,2) \geq \frac{4}{3} \cdot T_M(1,1)^2$$

This theorem solves Merino-Welsh conjecture for LPMs and characterizes the cases of equality.

Sketch of the proof

- *Base*: Proof for connected snakes.

Sketch of the proof

- *Base*: Proof for connected snakes.
- *Decomposition*: Each connected LPM is either a snake, or it has an element e for which both $M \setminus e$ and M/e are LPM.

Sketch of the proof

- *Base*: Proof for connected snakes.
- *Decomposition*: Each connected LPM is either a snake, or it has an element e for which both $M \setminus e$ and M/e are LPM.
- *Step-up*: Lemma that concludes the inequality for M from the inequality for $M \setminus e$ and M/e.

Sketch of the proof

- *Base*: Proof for connected snakes.
- *Decomposition*: Each connected LPM is either a snake, or it has an element e for which both $M \setminus e$ and M/e are LPM.
- *Step-up*: Lemma that concludes the inequality for M from the inequality for $M \setminus e$ and M/e.
- *Wrap-up*: We deal with non-connected LPM.

Base: Connected snakes

- The result without the $\frac{4}{3}$ factor has a proof using a result of Noble and Royle for series-parallel graphs.

Base: Connected snakes

- The result without the $\frac{4}{3}$ factor has a proof using a result of Noble and Royle for series-parallel graphs.
- We proceed by induction on the number of changes of direction of the snake. We solve 1 and 2 as inductive base.

Base: Connected snakes

- The result without the $\frac{4}{3}$ factor has a proof using a result of Noble and Royle for series-parallel graphs.
- We proceed by induction on the number of changes of direction of the snake. We solve 1 and 2 as inductive base.

$$\begin{aligned}
4 \cdot 3 \cdot (2^a - 1) &\geq 12 \cdot \left(1 + a + \frac{a(a-1)}{2} - 1\right) \\
&= 6a^2 + 6a = \frac{4}{3} \cdot (4a^2 + 4a) + \frac{2}{3} \cdot (a^2 + a) \\
&\geq \frac{4}{3} \cdot (2a+1)^2.
\end{aligned}$$

Base: Connected snakes

- The result without the $\frac{4}{3}$ factor has a proof using a result of Noble and Royle for series-parallel graphs.
- We proceed by induction on the number of changes of direction of the snake. We solve 1 and 2 as inductive base.

$$\begin{aligned}4 \cdot 3 \cdot (2^a - 1) &\geq 12 \cdot \left(1 + a + \frac{a(a-1)}{2} - 1\right) \\ &= 6a^2 + 6a = \frac{4}{3} \cdot (4a^2 + 4a) + \frac{2}{3} \cdot (a^2 + a) \\ &\geq \frac{4}{3} \cdot (2a+1)^2.\end{aligned}$$

- We establish the inductive step using a recursive formula for the number of valid paths.

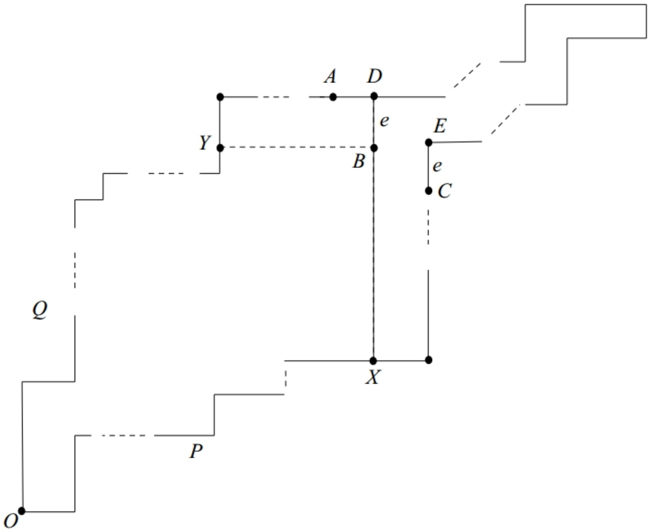
Decomposition lemma

Proposition

Let M be a connected LPM. Then

- *M is a snake or*
- *M has an element e for which both $M \setminus e$ and M/e are connected LPMs different from the trivial snake.*

Decomposition lemma



Step-up lemma

Lemma

Let M be a loopless and coloopless matroid and e an element of the ground set. If the desired inequality holds for $M \setminus e$ and M/e, then it also holds for M.

Step-up lemma

Lemma

Let M be a loopless and coloopless matroid and e an element of the ground set. If the desired inequality holds for $M \setminus e$ and M/e, then it also holds for M.

$$a = T_{M\setminus e}(2,0),\ b = T_{M\setminus e}(0,2),\ c = T_{M\setminus e}(1,1)$$
$$d = T_{M/e}(2,0),\ e = T_{M/e}(0,2),\ f = T_{M/e}(1,1)$$

Step-up lemma

Lemma

Let M be a loopless and coloopless matroid and e an element of the ground set. If the desired inequality holds for $M \setminus e$ and M/e, then it also holds for M.

$$a = T_{M\setminus e}(2,0),\ b = T_{M\setminus e}(0,2),\ c = T_{M\setminus e}(1,1)$$
$$d = T_{M/e}(2,0),\ e = T_{M/e}(0,2),\ f = T_{M/e}(1,1)$$

$$(a+d)(b+e) \geq \left(\sqrt{ab}+\sqrt{de}\right)^2 \geq \frac{4}{3}\cdot(c+f)^2.$$

Wrap up and non-connected LPMs

- We proceed again by induction, this time on the number of elements in the ground set.

Wrap up and non-connected LPMs

- We proceed again by induction, this time on the number of elements in the ground set.
- We use the decomposition lemma to go down.

Wrap up and non-connected LPMs

- We proceed again by induction, this time on the number of elements in the ground set.
- We use the decomposition lemma to go down.
- We use the inductive hypothesis and the step-up lemma to go up.

Wrap up and non-connected LPMs

- We proceed again by induction, this time on the number of elements in the ground set.
- We use the decomposition lemma to go down.
- We use the inductive hypothesis and the step-up lemma to go up.
- To deal with non-connected LPM, we apply what we know to each connected component and we use that the Tutte polynomial is multiplicative on direct sums of matroids.

Wrap up and non-connected LPMs

- We proceed again by induction, this time on the number of elements in the ground set.
- We use the decomposition lemma to go down.
- We use the inductive hypothesis and the step-up lemma to go up.
- To deal with non-connected LPM, we apply what we know to each connected component and we use that the Tutte polynomial is multiplicative on direct sums of matroids.

$$\begin{aligned} T_M(2,0) \cdot T_M(0,2) &= \prod_{i=1}^{n} a_i \cdot \prod_{i=1}^{n} b_i = \prod_{i=1}^{n} (a_i \cdot b_i) \\ &\geq \frac{4}{3} \cdot \prod_{i=1}^{n} c_i^2 = \frac{4}{3} \cdot \left(\prod_{i=1} c_i \right)^2 = \frac{4}{3} \cdot T_M(1,1)^2. \end{aligned}$$

Corollary: Merino-Welsh conjecture for LPMs

Theorem

Let M be a loopless and coloopless LPM. Then

$$T_M(2,0) \cdot T_M(0,2) \geq T_M(1,1)^2.$$

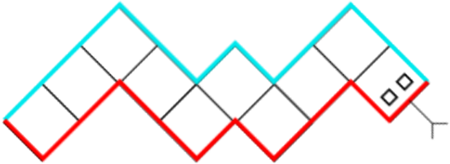
The equality holds if and only if M is a direct sum of trivial snakes. Otherwise, the right-hand side can be sharpened by a multiplicative factor of $\frac{4}{3}$.

Thanks

Thank you for your attention!

Thanks

Thank you for your attention!



`leomtz@im.unam.mx` - `http://blog.nekomath.com`